REDUCING THE POINCARÉ SERIES OF LOCAL RINGS TO THE CASE OF QUADRATIC RELATIONS

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Let $(R,\mathfrak{m},k)$ be a commutative, noetherian local ring and let M be a finitely generated R-module. By the Poincaré series of M we mean the formal power series

$$P_R^M = \sum_{p\geq 0} \dim_k \operatorname{Tor}_q^R(M,k)t^q$$

For a given M it is usually extremely difficult to compute P_R^M and in all cases where an explicit formula has been obtained, P_R^M turns out to be a rational function. Whether P_R^M is rational in general is still not known, but many attempts have been made to reduce the general case to hopefully simpler cases. One such reduction appeared in [1] where it was shown that the rationality of P_R^M for all R and M follows if P_R^k is rational for all R. Then Levin [2] made a beautiful reduction showing that it suffices to consider artinian local rings. In the present note we shall make a reduction to the case where the ring is defined by certain quadratic relations.

In the following let R_0 be a field or a complete regular local ring of dimension 1, whose maximal ideal is generated by a prime number.

DEFINITION 1. We will say that a local ring is defined by <u>special quadratic relations</u> if it has the form

$$R_0[[X_1,\ldots,X_m,Y_1,\ldots,Y_m]]/I$$

where I is an ideal generated by the quadratic forms

$$\sum_{i,j}\alpha_{ij}X_iY_j\ , \qquad \alpha_{ij}\in R_0$$

where the $m\times m$-matrices (α_{ij}) run through a set of matrices which is closed with respect to transposition.

DEFINITION 2. Let J be an ideal in a local ring R^* and put $R = R^*/J$. J will be called a <u>large</u> ideal if the canonical homomorphism $f:R^*\to R$ is large in the sense of Levin [3], i.e. if the induced map

$$\mathrm{Tor}^{R^*}(k,k)\to\mathrm{Tor}^{R}(k,k)$$

is surjective, k being the residue field of R. Observe that the maximal ideal in a local ring is always a large ideal.

PROPOSITION (Levin). Let $f:R^*\to R$ be a surjective homomorphism of local rings. Then the following statements are equivalent:

(i) $f:R^*\to R$ is a large homomorphism.

(ii) The canonical map $\mathrm{Tor}^{R^*}(R,k)\to\mathrm{Tor}^{R^*}(k,k)$ is injective.

(iii) $$P^M_{R^*} = P^R_{R^*}P^M_R$$
for all finitely generated R-modules M, considered as R^*-modules via f.

PROOF: See Theorem 1.1 in [3].

We are now in position to state the result of the present note.

THEOREM. If the Poincaré series P_{R*}^{J} is rational for all local rings R^* defined by special quadratic relations, and all large ideals J in R^*, then P_R^M is rational for all local rings R and all finitely generated R-modules M.

PROOF: To prove the rationality of P_R^M for all R and M it suffices to prove the rationality of P_R^k for all artinian R. This reduction follows from Theorem 3 in [1] combined with Theorem 1 in [2]. Thus, in the following R is assumed to be artinian. The idea of the proof is now to construct a large homomorphism $f:R^* \twoheadrightarrow R$ where R^* is a local ring defined by special quadratic relations.

By Cohens structure theorem R is an algebra over a ring R_0 with properties described above, and such that the structure map $R_0 \to R$ induces an isomorphism between the residue class fields of R_0 and R. Let $v_1,\dots,v_m$ be a set of generators for $\mathfrak{m}$, considered as an R_0-module, and consider the following set of $m\times m$-matrices (α_{ij}) with entries in R_0:

$$A = \{(\alpha_{ij}) \mid \sum_{i,j} \alpha_{ij}\, v_i v_j = 0\}$$

Now put

$$R^* = R_0[[X_1,\dots,X_m,Y_1,\dots,Y_m]]/I$$

where I is the ideal generated by all the quadratic forms $\Sigma\alpha_{ij}\, X_i Y_j$ where $(\alpha_{ij}) \in A$. Let $\bar{X}_i$ (resp. $\bar{Y}_i$) be the image of X_i (resp. Y_i) in R^* and let $f:R^* \twoheadrightarrow R$ be the unique homomorphism extending the structure map $R_0 \to R$ and sending $\bar{X}_i$ and $\bar{Y}_i$ to v_i.

According to the Proposition we will now show that f is a large homomorphism by showing that

$$(1) \qquad P_{R*}^{M} = P_{R*}^{R}\, P_R^M$$

for all finitely generated R-modules. Let F be a minimal R-free resolution of M. We shall first show that F can be lifted to an R^*-free complex F^* whose differential has coefficients in $\mathfrak{m}^*$, the maximal ideal of R^*.

For each homogeneous component F_q of F we select a basis of F_q as a free R-module. Let D_q be the matrix associated with the differential $F_q \to F_{q-1}$. We now have to lift D_q to matrices D_q^* with entries in $\mathfrak{m}^*$ in such a way that $D_{q-1}^* D_q^* = 0$. We do this in the following way. Observe that since F is a minimal resolution, each entry of D_q is in $\mathfrak{m}$, so for each entry c we can fix elements $\alpha_1, \dots, \alpha_m$ in R_0 depending on c such that

$$c = \sum_i \alpha_i v_i$$

To obtain D_q^* we replace each entry c by c^* where $c^* = \Sigma_i \alpha_i \bar{X}_i$ if q is odd, and $c^* = \Sigma_i \alpha_i \bar{Y}_i$ if q is even.
Then clearly the entries of the product $D_{q-1}^* D_q^*$ are quadratic expressions of the type $\Sigma \alpha_{ij} \bar{X}_i \bar{Y}_j$ where $\alpha_{ij} \in R_0$. We have

$$f(\Sigma \alpha_{ij} \bar{X}_i \bar{Y}_j) = \Sigma \alpha_{ij} v_i v_j$$

On the other hand $f(\Sigma \alpha_{ij} \bar{X}_i \bar{Y}_j)$ is zero since it is an entry of $D_{q-1} D_q = 0$. This shows that

$$(\alpha_{ij}) \in A, \text{ so } \Sigma \alpha_{ij} \bar{X}_i \bar{Y}_j = 0, \text{ which means that } D_{q-1}^* D_q^* = 0 .$$

The existence of the lifted complex F^* is now established.

To establish (1) let Y be a mimimal R^*-free resolution of R. Then by a standard spectral sequence argument the total complex $F^* \otimes_{R^*} Y$ is acyclic, so it is a mimimal R^*-free resolution of M. Hence

$$\begin{aligned} \mathrm{Tor}^{R^*}(M,k) &\simeq F^* \otimes Y \otimes k \simeq (F^* \otimes k) \otimes (Y \otimes k) \\ &\simeq (F \otimes k) \otimes (Y \otimes k) \simeq \mathrm{Tor}^R(M,k) \otimes \mathrm{Tor}^{R^*}(R,k) . \end{aligned}$$

So (1) follows.

Let us now assume that $P^J_{R^*}$ is rational for all large ideals J. Chosing $J = \mathfrak{m}^*$ gives the rationality of $P^k_{R^*}$, and chosing $J = \operatorname{Ker} f$ gives the rationality of $P^R_{R^*}$. Now substituting $M = k$ in (1) gives

$$P^k_R = P^k_{R^*}(P^R_{R^*})^{-1}$$

so P^k_R is rational. □

REMARKS. The homomorphism $f: R^* \to R$ has a few nice properties which we state without proof.

1) Any R-free complex Z which is bounded below, i.e. $Z_p = 0$ for all p sufficiently small, can be lifted to an R^*-free complex.

2) There exists a minimal R^*-algebra resolution of R. By a result due to Avramov and Rahbar-Rochandel this is in fact true for any surjective large homomorphism of local rings. Cf. Theorem 2.5 in [3].

3) The integer m in the definition of R^* can be chosen to be the length of $\mathfrak{m}$. In that case the ideal I can be generated by less than or equal to m^2 elements. (This comes from the fact that R_0 is a principal ideal ring and A is an R_0-submodule of the free R_0-module of all $m\times m$-matrices with entries in R_0.) That this estimate is "best possible" is clear from the following example:

$$R = k[t_1,\dots,t_n]\big/(t_1,\dots,t_n)^2$$

chose $m = n$. Then we have

$$R^* = k[[X_1,\dots,X_m,Y_1,\dots,Y_m]]\big/(\dots,X_iY_j,\dots) \qquad 1 \le i \le m,\ 1 \le j \le m.$$

REFERENCES

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3. ———, Large homomorphisms of local rings. To appear.